PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The Complete System of the Periods of a Hollow Vortex Ring

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Phil. Trans. R. Soc. Lond. A 1895 **186**, 603-619
doi: 10.1098/rsta.1895.0017

XVII. *The Complete System of the Periods of a Hollow Vortex Ring.*

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Communicated by JOSEPH LARMOR, *F.R.S.*

Received April 23,—Read May 16, 1895.

1. ACCORDING to the vortex theory of matter, atoms consist of vortex rings in an infinite perfect liquid, the æther. These rings may be either hollow or filled with rotating liquid. The cross section of the hollow or rotating core is in the simplest case small and the ring is circular. Such vortices have been investigated. It has been shown that they can exist, and that they are stable for certain types of deformation. In this paper the stability of the hollow vortex ring is investigated further, with a view to proving that it is stable for all small deformations of its surface. An attempt is also made to make the vortex theory of matter agree with the kinetic theory of gases as regards the relation between the velocity and the energy of an atom. On the latter theory the energy of an atom varies as the square of its velocity, while on the former theory the energy decreases as the velocity increases. As the two theories differ on a fundamental point, while the consequences of the kinetic theory agree over a wide range with experiment, those of the vortex theory are likely to be in discrepancy therewith. However, no account has been taken of the electric change which an atom must hold if electrolysis is to be explained. This electrification will evidently alter the relation between the energy and the velocity. The nature of the change thus produced is here discussed for the case of a hollow vortex, the surface of which behaves as a conductor of electricity, a representation which is dynamically realised by the theory of a rotationally-elastic fluid æther developed in Mr. LARMOR's paper,* "A Dynamical Theory of the Electric and Luminiferous Medium." The small oscillations also are worked out with a view to the discussion of the stability of an electrified vortex.

2. The velocity of translation of the vortex in its steady motion is constant and perpendicular to its plane. By impressing on the whole liquid a velocity equal and opposite to this, the hollow is reduced to rest. Since the cross section of the hollow is small, any small length of it may be regarded as cylindrical. A cylindrical vortex must, by reason of symmetry, have its cross section a circle, so that the cross section

* J. LARMOR, 'Phil. Trans.,' 1894, A, p. 719.

of the hollow of the annular vortex is approximately circular, and the hollow itself approximately a tore.

3. The motion of the fluid is referred to the toroidal coordinates u, v, w, where u, v are given in terms of the cylindrical coordinates ϖ, z, by the equation

$$u + \iota v = \log \frac{\varpi + \iota z + a}{\varpi + \iota z - a} \quad \ldots \ldots \ldots \ldots (1),$$

and w is the angular cylindrical coordinate.

The constant a and the position of the axes are so chosen that the hollow is approximately given by $u = \text{const.}$

On solving (1) for $z + \iota\varpi$, and equating real and imaginary parts,

$$\varpi = a \frac{\sinh u}{\cosh u - \cos v} = a \frac{\mathrm{S}}{\mathrm{C} - c} \quad \ldots \ldots \ldots \ldots (2),$$

$$z = -a \frac{\sin v}{\cosh u - \cos v} = -a \frac{s}{\mathrm{C} - c} \quad \ldots \ldots \ldots \ldots (3),$$

where C, c, S, s are written for $\cosh u$, $\cos v$, $\sinh u$, $\sin v$ respectively.

The square root of the Jacobian of u, v with respect to ϖ, z is ξ where

$$\xi^2 = \frac{1}{\left(\dfrac{d\varpi}{du}\right)^2 + \left(\dfrac{d\varpi}{dv}\right)^2} = \frac{(\cosh u - \cos v)^2}{a^2},$$

or,

$$\xi = \frac{\mathrm{C} - c}{a}.$$

If k be put for e^{-u}, k is small near the surface of the hollow, and the values of ϖ, z, ξ are

$$\left.\begin{aligned} \varpi &= a\,(1 + 2k \cos v) \\ z &= 2ak \sin v \\ \xi &= \frac{1 - 2k \cos v}{2ka} \end{aligned}\right\} \quad \ldots \ldots \ldots \ldots (4),$$

in which expression the squares and higher powers of k have been neglected.

4. The motion of the liquid, being symmetrical about the axis of the ring, may be expressed in terms of STOKES' current function ψ. Now ψ consists of two terms. One is $-\frac{1}{2}\mathrm{V}\varpi^2$, corresponding to a uniform motion of the liquid with velocity V. The other vanishes at an infinite distance from the ring and, therefore, can be expressed

in the form* $\{2(C - c)\}^{-\frac{1}{2}}\Sigma A_n R_n \cos nv$, where $R_n = S \frac{dP_n}{du}$ and P_n is that toroidal function of order n, which is finite when $u = 0$.

Thus,

$$\psi = -\tfrac{1}{2}V\varpi^2 + \frac{1}{\sqrt{2(C-c)}}\Sigma A_n R_n \cos nv \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

The first term under the sign Σ corresponds to a circulation about the critical circle $k = 0$. The succeeding terms correspond partly to the disturbance of this circulation by the free surface, and partly to the disturbance of the uniform flow $\psi = -\frac{1}{2}V\varpi^2$ by the same surface. Now since the annulus is nearly coincident with a flow surface of the circulation, and since V, as is easily seen from elementary considerations, is small, these terms are small. Their products by small quantities may therefore be neglected.

Hence, expanding by (4),

$$\psi = -\tfrac{1}{2}Va^2(1 + 4k\cos v) + (1 + k\cos v)k^{\frac{1}{2}}\{A_0R_0 + A_1R_1\cos v + \text{etc}\}. \quad (6).$$

Since $k = b$ is a flow surface, $\psi =$ const. when $k = b$, or

$$\text{const.} = A_0R'_0b^{\frac{1}{2}} - \tfrac{1}{2}Va^2 + (A_1R'_1b^{\frac{1}{2}} + A_0R'_0b^{\frac{3}{2}} - 2Va^2b)\cos v + A_2R'_2b^{\frac{1}{2}}\cos 2v + \text{etc.},$$

where R'_s means the value of R_s when k is put equal to b.

Hence, equating to zero the coefficients of $\cos v$, $\cos 2v$, etc.,

$$A_2 = A_3 = 0, \qquad A_1 = \frac{2Va^2b^{\frac{1}{2}} - A_0R'_0b}{R'_1}.$$

Also†

$$R_0 = -(\tfrac{1}{2}L - 1)k^{-\frac{1}{2}}, \qquad R_1 = \tfrac{1}{2}k^{-\frac{3}{2}},$$

where $L = \log 4/k$, whence, substituting,

$$A_1 = 4Va^2b^2 + (L' - 2)b^2.$$

Substituting the value of A_1 in (6),

$$\psi = -\tfrac{1}{2}Va^2 - A_0\frac{L-2}{2} + \left\{\frac{2Va^2b^2}{k} - 2Va^2k + A_0\frac{L'-2}{2k}b^2 - A_0\frac{L-2}{2}k\right\}\cos v \quad (7).$$

The fluid circulation is the line integral of the velocity taken along any curve

* W. M. Hicks, 'Phil. Trans.,' 1884.

† W. M. Hicks, 'Phil. Trans.,' 1884, p. 172.

enclosing the hollow. Taking the curve to be $k = b$, and putting for the velocity its value $\frac{\xi}{\varpi}\frac{d\psi}{du}$, we have

$$\text{circulation} = \mu = \int_0^{2\pi} \frac{dv}{\xi}\frac{\xi}{\varpi}\frac{d\psi}{du} = -\int_0^{2\pi} dv \frac{\mathrm{C} - \cos v}{\mathrm{S}a} b \frac{d\psi}{dk}$$
$$= -\int_0^{2\pi} dv \frac{\mathrm{C} - \cos v}{\mathrm{S}a} b \left[\frac{\mathrm{A}_0}{2b} - \left\{4\mathrm{V}a^2 + \mathrm{A}_0(\mathrm{L}' - 2) - \frac{\mathrm{A}_0}{2}\right\}\cos v\right]$$
$$= -\pi \mathrm{A}_0/a,$$

neglecting quantities of the second order.

Hence

$$\mathrm{A}_0 = -\mu a/\pi \quad \dots \dots \dots \dots \quad (8).$$

5. The motion of the liquid is now completely expressed in terms of μ and V by (7) and (8). We proceed to find the energy of the fluid motion and, with the object of writing down the condition that is satisfied at the free surface, the pressure of the liquid at the surface.

If ψ' is the current function that gives the motion of the liquid when the reversed velocity V is not impressed on the liquid, so that $\psi' = \frac{1}{2}\mathrm{V}\varpi^2 + \psi$, the energy of the actual motion is

$$\tfrac{1}{2}\rho \iint 2\pi\varpi\, d\varpi\, dz \frac{1}{\varpi^2}\left\{\left(\frac{d\psi'}{d\varpi}\right)^2 + \left(\frac{d\psi'}{dz}\right)^2\right\}$$
$$= \pi\rho\left[\int ds \frac{\psi'}{\varpi}\frac{d\psi'}{dn} - \iint d\varpi\, dz\, \psi'\left\{\frac{d}{d\varpi}\left(\frac{1}{\varpi}\frac{d\psi'}{d\varpi}\right) + \frac{1}{\varpi}\frac{d^2\psi'}{dz^2}\right\}\right],$$

by integration by parts, where, in the double integrals, ϖ and z are to take all values corresponding to space occupied by the liquid, and in the single integral, ds is an element of length of the cross-section of the boundary, dn the element of normal to the boundary, taken as positive when drawn from the liquid, and the integral is taken round the cross-section of the boundary. The double integral in the latter expression vanishes by the equation satisfied by ψ in an irrotational motion, and thus

$$\text{energy} = \pi\rho\int ds \frac{\psi'}{\varpi}\frac{d\psi'}{dn} = \pi\rho\int_0^{2\pi}\frac{dv}{\xi}\frac{\psi'}{\varpi}\xi\frac{d\psi'}{du},$$
$$= -\pi\rho\int_0^{2\pi} dv\, b\frac{1 - 2b\cos v}{a}\left\{\frac{\mathrm{A}_0}{2b} - \left(2\mathrm{V}a^2 + \mathrm{A}_0(\mathrm{L}' - 2) - \frac{\mathrm{A}_0}{2}\right)\cos v\right\}$$
$$\left\{-\mathrm{A}_0\frac{\mathrm{L}' - 2}{2} + 2\mathrm{V}a^2 b\cos v\right\}$$
$$= \frac{\pi^2\rho \mathrm{A}_0^2(\mathrm{L}' - 2)}{2a} = \tfrac{1}{2}\rho a\mu^2(\mathrm{L}' - 2) \quad \dots \dots \dots \dots \quad (9),$$

by (8).

If the pressure at any point of the liquid is Π, and at infinity is Π_∞,

$$\Pi = \Pi_\infty - \tfrac{1}{2}\rho q^2,$$

where q is the resultant velocity.

The components of the velocity perpendicular to the surfaces v = const. and u = const. are $\frac{\xi}{\varpi}\frac{d\psi}{du}$, $\frac{\xi}{\varpi}\frac{d\psi}{dv}$, respectively. But $\frac{d\psi}{dv} = 0$ on the flow surface $k = b$. Hence

$$q^2 = \frac{\xi^2}{\varpi^2}\left(\frac{d\psi}{du}\right)^2 = \frac{(\mathrm{C} - c)^4}{a^4\mathrm{S}^2}\,k^2\left(\frac{d\psi}{dk}\right)^2,$$

$$= \frac{1 - 8k\cos v}{4a^4}\left[\frac{\mathrm{A}_0}{2k} - \left\{\frac{2\mathrm{V}a^2b^2}{k^2} + \mathrm{A}_0\frac{(\mathrm{L}' - 2)\,b^2}{2k^2} + \mathrm{A}_0\frac{\mathrm{L} - 2}{2} - \frac{\mathrm{A}_0}{2} + 2\mathrm{V}a^2\right\}\cos v\right]^2,$$

or, putting $k = b$,

$$q^2 = \frac{\mathrm{A}_0{}^2}{16a^4b^2}\left\{1 - 4b\left(\mathrm{L}' - \tfrac{1}{2} - \frac{4\mathrm{V}a^2}{\mathrm{A}_0}\right)\cos v\right\},$$

$$= \frac{\mu^2}{16\pi^2a^2b^2}\left\{1 - 4b\left(\mathrm{L}' - \tfrac{1}{2} - \frac{4\pi\mathrm{V}a}{\mu}\right)\cos v\right\},$$

and thus

$$\Pi = \Pi_\infty - \frac{\mu^2\rho}{32\pi^2a^2b^2}\left\{1 - 4b\left(\mathrm{L}' - \tfrac{1}{2} - \frac{4\pi\mathrm{V}a}{\mu}\right)\cos v\right\} \quad . \quad . \quad . \quad (10).$$

The pressure is zero at the free surface, so that

$$\frac{\mu^2\rho}{32\pi^2a^2b^2} = \Pi\ , \qquad \frac{4\pi\mathrm{V}a}{\mu} = \mathrm{L}' - \tfrac{1}{2},$$

or

$$\mathrm{V} = \frac{\mu}{4\pi a}\left\{\log\frac{16\pi a\sqrt{(2\Pi_\infty)}}{\mu\sqrt{\rho}} - \tfrac{1}{2}\right\},$$

and thus V diminishes with increase of a.

6. To find the small oscillations we assume that the equation to the surface of the hollow when it vibrates is at time t

$$k = b\,\{1 + \beta e^{\iota(nv + mw + pt)}\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$$

where $\iota = \sqrt{(-1)}$ and m and n are positive integers, β being small. We then determine the corresponding disturbed motion of the liquid, and finally write down the condition that the pressure at the surface of the liquid is zero. If the resulting equation can be satisfied by any value of p, the assumption as to the form of the disturbed surface is justified, and $2\pi/p$ is the period of the oscillation. The displacement of the surface given by (11) is complex. Noting, however, that the signs of ι and of w may be changed, other possible equations to the surface are seen to be

$$k = b\{1 + \beta e^{\iota(-nv+mw-pt)}\},$$
$$k = b\{1 + \beta e^{\iota(nv-mw+pt)}\},$$
$$k = b\{1 + \beta e^{\iota(-nv-mw-pt)}\},$$

in which equations p has the same value as it has in (11), and is here supposed to be real. The displacements, being small, may be added and subtracted, so that

$$\left.\begin{aligned} k &= b\{1 + \beta \sin(nv + pt) \sin mw\}, \\ k &= b\{1 + \beta \sin(nv + pt) \cos mw\}, \\ k &= b\{1 + \beta \cos(nv + pt) \sin mw\}, \\ k &= b\{1 + \beta \cos(nv + pt) \cos mw\}, \end{aligned}\right\} \quad \ldots \ldots \quad (12)$$

are possible equations for the surface of the hollow at time t. These equations no longer contain imaginaries, and thus they correspond to real oscillations. If m is zero, the disturbance of the surface is symmetrical about the axis of the vortex, and consists of trains of waves, the crests of which are circles parallel to the critical circle, progressing round the hollow. These vibrations will be called the fluted vibrations of the vortex. If n is zero, the disturbance consists of alternate swellings and contractions of the hollow disposed at equal intervals along its length. These vibrations will be called the beaded vibrations of the vortex. As given by (12) they are standing waves. On adding or subtracting the disturbances given by the first and last or the second and third of these equations, beaded vibrations are obtained which progress along the length of the hollow. If both m and n are zero, the disturbance consists of a uniform periodic enlargement of the vortex. This vibration will be called the pulsation of the vortex. If n is unity, the disturbance consists of a displacement of the axis of the hollow, without alteration of its cross section. These vibrations will be called the sinuous vibrations of the vortex.

7. The motion of the liquid is that which subsists when there is no vibration, together with small terms introduced by the vibration. The former has already been expressed in terms of a stream function ψ given by (7). Let the corresponding velocity potential be ϕ_1. The terms due to the vibration must have an acyclic velocity potential. Let this potential be ϕ_2. Since ϕ_2 vanishes at infinity it can be expressed in a series of the form* $D\sqrt{2(C-c)}\ {}_mP_n\ e^{\iota(nv+mw+pt)}$, where ${}_mP_n$ is the associated toroidal function of order n and rank m, so that

* W. M. HICKS, 'Phil. Trans.,' 1881.

$$ {}_m\mathrm{P}_n = \int_0^\pi \frac{\cos m\theta \, d\theta}{(\mathrm{C} - \mathrm{S}\cos\theta)^{(2n+1)/2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13). $$

We shall assume, our assumption to be justified by the result, that ϕ_2 reduces to the single term $\mathrm{D}\sqrt{2(\mathrm{C}-c)}\;{}_m\mathrm{P}_n\; e^{\iota(nv+mw+pt)}$. Since D is small, we may neglect all but the lowest powers of k in this expression, and thus

$$ \phi_2 = \mathrm{D}\,\frac{{}_m\mathrm{P}_n}{\sqrt{k}}\, e^{\iota(nv+mw+pt)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14). $$

The value of D must now be found from the condition that the velocity of the free surface normal to itself is equal to the velocity of the liquid normal to the surface. The velocity of the surface normal to itself is

$$ -\frac{1}{\xi}\frac{du}{dt} = \frac{1}{\xi k}\frac{dk}{dt} $$

$$ = 2pab\iota\beta e^{\iota(nv+mw+pt)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15) $$

from (11).

The components of the velocity due to ψ are;

perpendicular to $u = $ const.,

$$ \frac{\xi}{\varpi}\frac{d\psi}{dv} = 0 \text{ to our order}; $$

perpendicular to $v = $ const.,

$$ \frac{\xi}{\varpi}\frac{d\psi}{du} $$

$$ = -\frac{1-4k\cos v}{2a^2k}\,k\left\{\frac{\mathrm{A}_0}{2k} + \left(\frac{\mathrm{A}_0}{2} - \mathrm{A}_0\,\frac{\mathrm{L}-2}{2} - \mathrm{A}_0\,\frac{\mathrm{L}'-2}{2}\,\frac{b^2}{k^2} - 2\mathrm{V}a^2 - \frac{2\mathrm{V}a^2b^2}{k^2}\right)\cos v\right\}, $$

or at the surface given by (11),

$$ = -\frac{1-4b\cos v}{2a^2}\left\{\frac{\mathrm{A}_0}{2b} - \frac{\mathrm{A}_0}{2b}\,\beta e^{\iota(nv+mw+pt)} + \left(\frac{\mathrm{A}_0}{2} - \mathrm{A}_0(\mathrm{L}-2) - 4\mathrm{V}a^2\right)\cos v\right\} $$

$$ = \frac{\mu}{4\pi ab}\left\{1 - \beta e^{\iota(nv+mw+pt)} - 2b\left(\mathrm{L}' - \tfrac{1}{2} - \frac{4\pi\mathrm{V}a}{\mu}\right)\cos v\right\}; $$

perpendicular to $w = $ const.,

$$ 0 $$

(16).

The velocities here, as elsewhere, are considered to be positive if measured in the direction of increase of k, v, w.

The components of velocity given by ϕ_2 are;

perpendicular to $u=$ const.,

$$-\xi\frac{d\phi_2}{du}=\frac{\mathrm{D}}{2a}\frac{d}{db}\left(\frac{{}_m\mathrm{P}_n{}'}{\sqrt{b}}\right)e^{\iota(nv+mw+pt)};$$

perpendicular to $v=$ const.,

$$\xi\frac{d\phi_2}{dv}=\frac{\iota n\mathrm{D}}{2ab}\frac{{}_m\mathrm{P}_n{}'}{\sqrt{b}}e^{\iota(nv+mw+pt)};$$

perpendicular to $w=$ const.,

$$\frac{1}{\varpi}\frac{d\phi_2}{dw}=\frac{\iota m\mathrm{D}}{a}\frac{{}_m\mathrm{P}_n{}'}{\sqrt{b}}e^{\iota(nv+mw+pt)}. \quad \ldots \quad (17).$$

The direction cosines of the normal to the surface are

$$1,\quad -\iota n\beta e^{\iota(nv+mw+pt)},\quad -2\iota m\beta e^{\iota(nv+mw+pt)} \quad \ldots \ldots \quad (18).$$

Hence the velocity of the liquid normal to the surface is, from (16), (17), (18),

$$-\frac{\mu}{4\pi ab}\iota n\beta e^{\iota(nv+mw+pt)}+\frac{\mathrm{D}}{2a}\frac{d}{db}\left(\frac{{}_m\mathrm{P}_n{}'}{\sqrt{b}}\right)e^{\iota(nv+mw+pt)},$$

neglecting terms small compared with those retained.

This is equal to the velocity of the surface normal to itself, so that from (15),

$$-\frac{\mu}{4\pi ab}\iota n\beta e^{\iota(nv+mw+pt)}+\frac{\mathrm{D}}{2a}\frac{d}{db}\left(\frac{{}_m\mathrm{P}_n{}'}{\sqrt{b}}\right)e^{\iota(nv+mw+pt)}=2pab\iota\beta e^{\iota(nv+mw+pt)},$$

which gives

$$\mathrm{D}=2\left(\frac{\mu n}{4\pi}+2pa^2b^2\right)\frac{\iota\beta}{b\dfrac{d}{db}\left(\dfrac{{}_m\mathrm{P}_n{}'}{\sqrt{b}}\right)} \quad \ldots \ldots \ldots \quad (19).$$

Substituting from this result and (16) and (17), the components of the velocity of the liquid at the surface of the hollow become;

perpendicular to $u=$ const.,

$$\frac{1}{ab}\left(\frac{\mu n}{4\pi}+2pa^2b^2\right)\iota\beta e^{\iota(nv+mw+pt)};$$

perpendicular to $v=$ const.,

$$\frac{\mu}{4\pi ab}\left\{1-\beta e^{\iota(nv+mw+pt)}-2b\left(\mathrm{L}'-\tfrac{1}{2}-\frac{4\pi\mathrm{V}a}{\mu}\right)\cos v\right\}-\frac{n\lambda}{ab}\left(\frac{\mu n}{4\pi}+2pa^2b^2\right)\beta e^{\iota(nv+mw+pt)};$$

perpendicular to $w =$ const.,

$$-\frac{m\lambda}{a}\left(\frac{\mu n}{4\pi} + 2pa^2b^2\right)\beta\,;$$

where

$$\lambda = \frac{{}_m\mathrm{P}_n'/\sqrt{b}}{b\,\dfrac{d}{db}\left(\dfrac{{}_m\mathrm{P}_n'}{\sqrt{b}}\right)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20).$$

The equation giving the pressure is

$$\Pi = \Pi_\infty - \tfrac{1}{2}\rho q^2 - \rho\,\frac{d}{dt}(\phi_1 + \phi_2).$$

Since ϕ_1 is independent of t, on substituting from (14) and (19), and equating the pressure to zero,

$$0 = \Pi = \Pi_\infty - \frac{\mu^2\rho}{32\pi^2a^2b^2}\left\{1 - 4b\left(\mathrm{L}' - \tfrac{1}{2} - \frac{4\pi\mathrm{V}a}{\mu}\right)\right.$$
$$\left. - 2\left(1 + n^2\lambda + \frac{8\pi a^2b^2n\lambda p}{\mu}\right)\beta e^{\iota(nv+mw+pt)}\right\} + \rho\lambda p\left(\frac{\mu n}{2\pi} + 4pa^2b^2\right)\beta e^{\iota(nv+mw+pt)} \;(21).$$

In this equation the terms not involving the exponential cancel by the equations defining the steady motion. The equation can therefore be satisfied by some value of p, which fact justifies the original assumption as to the nature of the disturbance of the surface. On dividing by $\beta e^{\iota(nv+mw+pt)}$ we find as equation for p,

$$\frac{\mu^2\rho}{16\pi^2a^2b^2}\left(1 + n^2\lambda + \frac{8\pi a^2b^2n\lambda p}{\mu}\right) + p\lambda\rho\left(\frac{\mu n}{2\pi} + 4pa^2b^2\right) = 0,$$

or

$$p^2 + 2pn\,\frac{\mu}{8\pi a^2b^2} + \left(n^2 + \frac{1}{\lambda}\right)\left(\frac{\mu}{8\pi a^2b^2}\right)^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad (22).$$

8. It now remains to find the value of λ, defined by (20).

Now

$${}_m\mathrm{P}_n' = \int_0^\pi \frac{\cos m\theta\, d\theta}{(\mathrm{C} - \mathrm{S}\cos\theta)^{(2n+1)/2}},$$

or, expressing in terms of k and remembering that k is small,

$${}_m\mathrm{P}_n' = (2k)^{(2n+1)/2}\int_0^\pi \frac{\cos m\theta\, d\theta}{\{1 - \cos\theta + k^2(1 + \cos\theta)\}^{(2n+1)/2}}$$
$$= (4k)^{(2n+1)/2}\int_0^\epsilon \frac{d\theta}{(\theta^2 + 4k^2)^{(2n+1)/2}} + (2k)^{(2n+1)/2}\int_\epsilon^\pi \frac{\cos m\theta\, d\theta}{(1 - \cos\theta)^{(2n+1)/2}} \quad . \quad . \quad (23),$$

where ϵ is small absolutely, but yet large compared with k. If n is not zero, the latter integral can be neglected in comparison with the former, and the limits of the former can be taken to be 0 and ∞, so that

$$ {}_mP_n = (4k)^{(2n+1)/2} \int_0^\infty \frac{d\theta}{(\theta^2 + 4k^2)^{(2n+1)/2}} . $$

Now

$$ \int_0^\infty \frac{d\theta}{(\theta^2+\alpha)^{(2n+1)/2}} = \frac{(-1)^{n-1}\left(\frac{d}{d\alpha}\right)^{n-1}}{\frac{3}{2}\cdot\frac{5}{2}\cdots\frac{2n-1}{2}} \int_0^\infty \frac{d\theta}{(\theta^2+\alpha)^{\frac{3}{2}}} $$

$$ \propto \left(\frac{d}{d\alpha}\right)^{n-1}\left[\frac{1}{\alpha}\frac{\theta}{\sqrt{\theta^2+\alpha}}\right]_0^\infty $$

$$ \propto \frac{1}{\alpha^n} . $$

Hence, in the case where n is not zero, ${}_mP_n$ varies as $k^{-(2n-1)/2}$ when k is small, so that

$$ \lambda = -1/n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24). $$

If n is zero, from (23)

$$ {}_mP_0 = (4k)^{\frac{1}{2}} \int_0^\epsilon \frac{d\theta}{\sqrt{\theta^2+4k^2}} + k^{\frac{1}{2}} \int_\epsilon^\pi \frac{\cos m\theta \, d\theta}{\sin(\theta/2)} $$

$$ = (4k)^{\frac{1}{2}} \log\frac{\epsilon}{k} + 2k^{\frac{1}{2}} \log\frac{4}{\epsilon} - k^{\frac{1}{2}} \int_0^\pi \frac{1-\cos m\theta}{\sin(\theta/2)} d\theta. $$

Calling the last integral γ_m we have

$$ \gamma_{m+1} - \gamma_m = 2\int_0^\pi d\theta \sin\frac{2m+1}{2}\theta = \frac{4}{2m+1} . $$

Also $\gamma_0 = 0$, therefore

$$ \gamma_m = 4\left(1 + \tfrac{1}{3} + \tfrac{1}{5} + \ldots + \frac{1}{2m-1}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad (25), $$

and

$$ {}_mP_0 = 2k^{\frac{1}{2}} \log\frac{4}{k} - k^{\frac{1}{2}}\gamma_m, $$

giving

$$ \lambda = -(L' - \tfrac{1}{2}\gamma_m) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26). $$

9. In the preceding paragraph, $\cos m\theta$ was put equal to unity in the integrals taken from 0 to ϵ. If m is so large that $m\epsilon$ is finite, this step is no longer valid. In this case, however, the second integral of (23) vanishes in comparison with the first, and the upper limit of the first may be taken to be infinity, so that

$$ {}_{m}\mathrm{P}_{n} = (4k)^{(2n+1)/2} \int_{0}^{\infty} \frac{\cos n\theta \, d\theta}{(\theta^2 + 4k^2)^{(2n+1)/2}} = \frac{2}{k^{(2n-1)/2}} \int_{0}^{\infty} \frac{\cos 2m\chi \, d\chi}{(1+\chi^2)^{(2n+1)/2}}, $$

and

$$ \frac{{}_{m}\mathrm{P}_{n}}{\sqrt{k}} = \frac{2}{k^n} \int_{0}^{\infty} \frac{\cos 2m\chi \, d\chi}{(1+\chi^2)^{(2n+1)/2}}, $$

a BESSEL's function of the second kind of order n with imaginary argument. Denoting this function by the symbol K_n,

$$ \frac{{}_{m}\mathrm{P}_{n}}{\sqrt{k}} \propto \mathrm{K}_{n}(2\,mk), \qquad \lambda = \frac{\mathrm{K}_n(2mb)}{2mb\mathrm{K}_n'(2mb)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27). $$

10. Substituting in (22) the value of λ given by (24), (26), (27), we have as equations for p,
if m is finite and n not zero,

$$ p^2 + 2pn\frac{\mu}{8\pi a^2 b^2} + n(n-1)\left(\frac{\mu}{8\pi a^2 b^2}\right)^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (28); $$

if m is finite and n zero,

$$ p^2 = \frac{1}{\mathrm{L}' - \frac{1}{2}\gamma_m}\left(\frac{\mu}{8\pi a^2 b^2}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (29); $$

if m is great, whatever n may be,

$$ p^2 + 2pn\frac{\mu}{8\pi a^2 b^2} + \left\{n^2 + \frac{2mb\mathrm{K}_n'(2mb)}{\mathrm{K}_n(2mb)}\right\}\left(\frac{\mu}{8\pi a^2 b^2}\right)^2 = 0 \quad . \quad . \quad . \quad (30). $$

The first of these equations agrees, in the case of $m = 0$, *i.e.*, in the case of fluted vibrations, with that obtained by Mr. BASSET,* and the second, in the case of $m = 0$, *i.e.*, in the case of pulsations, with that obtained by Mr. HICKS.† The first of these equations evidently gives in all cases real values of p. The second also does so, for L' is large and γ_m finite. The third also gives real values of p, for, by a property of the function K_n, easily proved from its definition as a definite integral, K_n is positive and its differential coefficient negative for real positive values of its argument. Hence all the values of p are real. One, however, given by the first of these equations when $n = 1$, is zero. This shows that the corresponding value of p is not, as are the other values of p, of the second order of large quantities, but of some lower order. These equations, then, show that the hollow vortex is stable for all displacements of its surface, with the possible exception of such as involve sinuous vibrations of the slow type. These latter vibrations have been investigated by Professor THOMSON,‡

* A. B. BASSET, 'Hydrodynamics,' vol. 2, p. 92.

† W. M. HICKS, 'Phil. Trans.,' 1884.

‡ J. J. THOMSON, 'Phil. Trans.,' 1882, and "Motion of Vortex Rings," Part I.

who proves that the vortex is stable for sinuous displacements, and finds the periods of the sinuous vibrations.

11. The electrified vortex. If the vortex carries a charge of electricity on its surface, the surface condition is altered. The effect of the electricity is to cause a tension to act on the surface which at any point is $F^2/8\pi$ per unit area, where F is the electric force. This tension must be balanced by the pressure of the liquid. Hence the surface condition is

$$\Pi = F^2/8\pi \quad \ldots \ldots \ldots \quad (31).$$

The electric state of the system is defined by the electric potential ϕ. Since ϕ vanishes at infinity, and is an even function of v, its expansion in a series of toroidal harmonics must be*

$$\phi = \sqrt{2(C - c)}\, \Sigma B_n P_n \cos nv \quad \ldots \ldots \ldots \quad (32).$$

The first term in the series is the potential, due to a uniform line distribution on the critical circle $k = 0$. If there were exact coincidence between the surface of the hollow and some equipotential of this distribution, the terms after the first would vanish. As an approximate coincidence subsists, these terms will be small, and their products by small quantities may be neglected.

Expanding (32),

$$\phi = B_0 P_0 k^{-\frac{1}{2}} + (B_1 k^{-\frac{1}{2}} P_1 - B_0 k^{\frac{1}{2}} P_0) \cos v + B_2 P_2 k^{-\frac{1}{2}} \cos 2v + \&c. \quad . \quad (33).$$

Putting $k = b$, ϕ is equal to the constant potential ϕ_0 of the vortex,

$$\phi_0 = B_0 P_0' b^{-\frac{1}{2}} + (B_1 b^{-\frac{1}{2}} P_1' - B_0 b^{\frac{1}{2}} P_0') \cos v + B_2 P_2 b^{-\frac{1}{2}} \cos 2v + \&c.,$$

so that

$$\phi_0 = B_0 P_0' b^{-\frac{1}{2}}, \quad B_1 = B_0 \frac{b P_0'}{P_1'}$$

$$B_2 = B_3 = \ldots = 0.$$

Also,† as

$$P_0 = 2Lk^{\frac{1}{2}}, \qquad P_1 = 2k^{-\frac{1}{2}},$$

to our order, we have

$$B_0 = \frac{\phi_0}{2L'}, \qquad B_1 = \frac{b^2}{2}\phi_0.$$

Substituting these values in (33),

$$\phi = \phi_0 \frac{L}{L'} + \phi_0 k \left(\frac{b^2}{k^2} - \frac{L}{L'}\right) \cos v.$$

* W. M. Hicks, 'Phil. Trans.,' 1881, p. 618.

† W. M. Hicks, 'Phil. Trans.,' 1884, p. 171.

The force normal to $u=$ const. is thus

$$-\xi k\frac{d\phi}{dk}=\frac{1-2k\cos v}{2ak}k\left\{\phi_0\frac{1}{kL'}+\phi_0\left(\frac{b^2}{k^2}-\frac{1}{L'}+\frac{L}{L'}\right)\cos v\right\}$$
$$=\frac{\phi_0}{2akL'}\left\{1+k\left(\frac{b^2}{k^2}L'+L-3\right)\cos v\right\}.$$

As the force normal to $v=$ const. vanishes on the equipotential $k=b$, the resultant force F is the value when $k=b$ of the above,

$$F=\frac{\phi_0}{2abL'}\{1+b(2L'-3)\cos v\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34).$$

The charge on the vortex is

$$E=\int_0^{2\pi}\frac{dv}{\xi}2\pi\varpi\frac{F}{4\pi},$$
$$=\int_0^{2\pi}\frac{dv}{2L'}\phi_0 a(1+4k\cos v)\{1+b(2L'-3)\cos v\},$$
$$=\frac{\pi a\phi_0}{L'},$$

or,

$$\phi_0=\frac{L'}{\pi a}E,$$

so that, substituting in (34),

$$F=\frac{E}{2\pi a^2 b}\{1+b(2L'-3)\cos v\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35).$$

The electric energy is

$$\tfrac{1}{2}\phi_0 E=\frac{L'}{2\pi a}E^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36).$$

12. We are now in a position to write down the equation giving the steady motion of an electrified vortex. Equating the value of Π, given by (10), to that of $F^2/8\pi$, given by (35), we have

$$\frac{E^2}{32\pi^3a^4b^2}\{1+2b(2L'-3)\cos v\}=\Pi_\infty-\frac{\mu^2\rho}{32\pi^2a^2b^2}\left\{1-4b\left(L'-\tfrac{1}{2}-\frac{4\pi Va}{\mu}\right)\cos v\right\},$$

which gives, on equating separately the terms involving v and those independent of v,

$$E^2+\pi\rho a^2\mu^2=32\pi^3a^4b^2\Pi_\infty \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (37),$$

$$V=\frac{\mu}{4\pi a}\left\{L'-\tfrac{1}{2}-\frac{E^2}{\pi\rho a^2\mu^2}(L'-\tfrac{3}{2})\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38).$$

The energy of the vortex is the sum of three terms. The first is the energy of the electricity, found in (36). The second is the kinetic energy of the liquid, found in (9). The third is the potential energy due to the presence of the hollow. Since the hollow could, if the liquid were brought to rest and the electricity removed, be filled by liquid flowing into it at pressure Π_∞, the last term is the product of Π_∞ and the volume of the hollow. Now, from (4), if k is small we have

$$(\varpi - a)^2 + z^2 = 4a^2k^2,$$

so that the radius of the cross-section of the ring is $2ab$, and the distance of the centre of the cross-section from the centre of the ring is a. Thus the volume of the hollow is $8\pi^2a^3b^2$, and the total energy on substituting for Π_∞ from (37) is

$$U = \tfrac{1}{2}\rho a\mu^2(L' - 2) + \frac{L'}{2\pi a}E^2 + 8\pi^2a^3b^2\,\frac{E^2 + \pi\rho a^2\mu^2}{32\pi^3a^4b^2}$$

$$= \frac{E^2}{2\pi a}(L' + \tfrac{1}{2}) + \frac{\rho a\mu^2}{2}(L' - \tfrac{3}{2}). \qquad (39).$$

Equations (37), (38), (39) give all desired information about the steady motion of an electrified vortex. The first effect of electrification is to diminish the velocity of the vortex. If $E^2/\pi\rho a^2\mu^2 = (L' - \frac{1}{2})/(L' - \frac{3}{2})$, V vanishes, while for values of E greater than that given by this equation, V is negative and the vortex moves backwards. This state of rest of the vortex may evidently be obtained as well by decreasing a as by increasing E. Hence an electrified vortex ring, however small the charge may be, will be reduced to rest if its radius is reduced sufficiently, and if its radius is further reduced, will move backwards, *i.e.*, in a direction opposed to that of the motion of the fluid at the centre of the ring.

Since

$$\frac{dU}{da} = \frac{\rho\mu^2}{2}(L' - \tfrac{1}{2}) - \frac{E^2}{2\pi a^2}(L' - \tfrac{3}{2}) = 2\pi a\mu V,$$

the energy of the vortex is least when its radius is such that it is stationary. If a is nearly equal to this value the difference between a constant and the energy of the vortex varies as the square of the velocity. This relation is only approximate and does not hold if a differs much from the value that makes $V = 0$. If a is so large that E^2 can be neglected in comparison with $\pi\rho a^2\mu^2$,

$$U = \tfrac{1}{2}\rho a\mu^2(L' - \tfrac{3}{2}), \qquad V = \mu(L' - \tfrac{1}{2})/4\pi a,$$

and as L′ may, if b is very small, be considered a large constant, $U \propto 1/V$ in this case. If, however, a is so small that $\pi\rho a^2\mu^2$ can be neglected in comparison with E^2,

$$U = E^2 (L' + \tfrac{1}{2})/2\pi a, \qquad V = E^2 (L' - \tfrac{3}{2})/4\pi^2 \rho a^3 \mu$$

numerically, and hence $U \propto \sqrt[3]{V}$. It seems, therefore, that the vortex theory of matter is not much improved in respect of the relations between the velocity and the energy by taking account of the atomic charge, unless the atom is supposed always to have nearly the radius corresponding to minimum energy.

[An electric charge can, of course, have no direct action on the velocity of the vortex. The manner of its action may be illustrated by considering what happens when a vortex ring is suddenly charged. The charge will produce two distinct effects. It will increase the cross-section of the hollow, and also its aperture. The former effect has no further consequences, but the motion of the fluid involved in the latter effect causes the velocity of the fluid at the front of the hollow to decrease, and that at the rear to increase. This causes an increase of pressure in front and a decrease behind, and so causes the ring to go more slowly.—15th August, 1895.]

[It may be desirable to point out that the alteration in the relation between the energy and the aperture of the ring, which is caused by an electric charge, takes place only in the potential part of that energy. The kinetic energy of the fluid is unaltered (to our order of approximation) by the electric charge.—August 29, 1895.]

13. In order to discuss the stability of an electrified vortex, the periods of the small oscillations must be found. For this purpose we must find the surface value of the electric force when the surface of the vortex is given by (11). The electric potential ϕ is now the sum of two terms. The first is the value that it has in the steady motion. The second we shall assume to be of the form

$$G \frac{{}_m P_n}{\sqrt{k}} e^{\iota (nv + mw + pt)},$$

so that

$$\phi = \phi_0 \frac{L}{L'} + k \left(\frac{b^2}{k^2} - \frac{L}{L'} \right) \phi_0 \cos v + G \frac{{}_m P_n}{\sqrt{k}} e^{\iota (nv + mw + pt)} \quad . \quad . \quad . \quad (40).$$

When

$$k = b \{ 1 + \beta e^{\iota (nv + mw + pt)} \}$$

this is to be constant. Hence

$$\text{const.} = \phi_0 - \frac{\phi_0}{L'} \beta e^{\iota (nv + mw + pt)} + G \frac{{}_m P_n'}{\sqrt{b}} e^{\iota (nv + mw + pt)}$$

giving

$$G = \frac{\phi_0}{L'} \frac{{}_m P_n'}{\sqrt{b}} \beta.$$

Equation (40) now gives

$$F = \xi k \frac{d\phi}{dk}$$

$$= \frac{1 - 2b \cos v}{2ab} b \left\{ \frac{\phi_0}{L'} \frac{1}{b} + \phi_0 \left(2 - \frac{1}{L'} \right) \cos v - \frac{\phi_0}{bL'} \beta e^{\iota (nv + mw + pt)} \right.$$

$$\left. - G \frac{d}{db} \left(\frac{{}_m P_n'}{\sqrt{b}} \right) e^{\iota (nv + mw + pt)} \right\}.$$

The total electric charge is the surface integral of $F/4\pi$.

The terms involving the exponential will evidently vanish on integration, except in the case when m and n are both zero, *i.e.*, except in the case of pulsations.

Hence, excluding pulsations, E has the same value as before, and

$$F = \frac{E}{2\pi a^2 b}\left\{1 + b(2L' - 3)\cos v - \left(1 + \frac{1}{\lambda}\right)\beta e^{\iota(nv+mw+pt)}\right\}.$$

The surface condition, given by (31) and (21), is

$$\begin{aligned}\Pi_\infty = {} & \frac{E^2}{32\pi^3 a^4 b^2}\left\{1 + 2b(2L' - 3)\cos v - 2\left(1 + \frac{1}{\lambda}\right)\beta e^{\iota(nv+mw+pt)}\right\} \\ & + \frac{\mu^2\rho}{32\pi^2 a^2 b^2}\left\{1 - 4b\left(L' - \tfrac{1}{2} - \frac{4\pi Va}{\mu}\right)\cos v - 2\left(1 + n^2\lambda + \frac{8\pi a^2 b^2 n\lambda p}{\mu}\right)\beta e^{\iota(nv+mw+pt)}\right\} \\ & - p\lambda\rho\left(\frac{\mu n}{2\pi} + 4pa^2b^2\right)\beta e^{\iota(nv+mw+pt)} \quad \ldots \quad (41).\end{aligned}$$

In this equation the terms not involving the exponential cancel by (37) and (38). The equation therefore can be satisfied by some value of p, which fact verifies our assumption as to the nature of the disturbance in this case. On simplification, the equation becomes

$$p^2 + 2pn\frac{\mu}{8\pi a^2 b^2} + \left\{n^2 + \frac{1}{\lambda} + \left(\lambda + \frac{1}{\lambda}\right)\frac{E^2}{\pi\rho a^2\mu^2}\right\}\left(\frac{\mu}{8\pi a^2 b^2}\right)^2 = 0 \quad . \quad . \quad (42).$$

If $m = n = 0$, this equation does not hold. In this case the equation to the surface is $k = b(1 + \beta e^{\iota pt})$. Now, from (35) the force on the surface of the tore $k = b$ is

$$\frac{F^2}{8\pi} = \frac{E^2}{32\pi^3 a^4 b^2}\quad\{1 + 2b(2L' - 3)\cos v\}.$$

Putting therefore $b + b\beta e^{\iota pt}$ in place of b, the force becomes

$$\frac{E^2}{32\pi^3 a^4 b^2}\{1 + 2b(2L' - 3)\cos v - 2\beta e^{\iota pt}\}.$$

Equating this to the pressure, we obtain

$$p^2 + \left(\frac{1}{\lambda} + \frac{1}{\lambda}\frac{E^2}{\pi\rho a^2\mu^2}\right)\left(\frac{\mu}{8\pi a^2 b^2}\right)^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43).$$

14. Substituting in (42) and (43) the values found in (24), (26), (27) for λ, we have as equations for p;

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if m is finite and n not zero,

$$p^2 + 2pn \frac{\mu}{8\pi a^2 b^2} + n(n-1)\left\{1 + \frac{E^2}{\pi\rho a^2\mu^2}\right\}\left(\frac{\mu}{8\pi a^2 b^2}\right)^2 = 0 \quad . \quad . \quad . \quad (44);$$

if m is finite and n zero,

$$p^2 = \left\{\frac{1}{L' - \frac{1}{2}\gamma_m} + \frac{(L' - \frac{1}{2}\gamma_m)^2 + 1}{L' - \frac{1}{2}\gamma_m}\frac{E^2}{\pi\rho a^2\mu^2}\right\}\left(\frac{\mu}{8\pi a^2 b^2}\right)^2 \quad . \quad . \quad . \quad . \quad (45);$$

if m and n are both zero,

$$p^2 = \frac{1}{L' - \frac{1}{2}\gamma_m}\left(1 + \frac{E^2}{\pi\rho a^2\mu^2}\right)\left(\frac{\mu}{8\pi a^2 b^2}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46);$$

if m is large, whatever n may be,

$$p^2 + 2pn \frac{\mu}{8\pi a^2 b^2} + \left[n^2 + \frac{2mb K_n'(2mb)}{K_n(2mb)} + \frac{\{2mb K_n'(2mb)\}^2 + \{K_n(2mb)\}^2}{2mb K_n'(2mb) K_n(2mb)} \frac{E^2}{\pi\rho a^2\mu^2}\right]\left(\frac{\mu}{8\pi a^2 b^2}\right)^2 = 0 \quad . \quad . \quad (47).$$

The values of p given by (45) and (46) are always real, but those given by (44) and (47) are imaginary for sufficiently large values of n. Taking (44), if p is to be real, we must have $n - (n^2 - n)\dfrac{E^2}{\pi\rho a^2\mu^2}$ positive. If $E^2/\pi\rho a^2\mu^2 = (L' - \frac{1}{2})/(L' - \frac{3}{2})$, the condition that $V = 0$, n must therefore not be greater than 1 for stability, so that all the fluted vibrations of the stationary vortex are unstable.

Thus the presence of an electric charge on the vortex makes it unstable, however small the charge may be. The modification of the vortex-atom theory of matter here considered is therefore not admissible unless some alteration in the hypothesis be made which will convert the instability of the vortex into stability.